From the figure, $z_k''\Delta A < \Delta V < z_k'''\Delta A$, where z_k''' and z_k'' are respectively maximum and minimum values of z in ΔV.

Then

$$z_k'' < z_k' < z_k''', \tag{2}$$

$$\Delta V = z_k'\left(\rho + \frac{\Delta\rho}{2}\right)\Delta\theta\Delta\rho, \tag{3}$$

whence, applying the generalized form of Theorem 2*a* to (1), (2) and (3),

$$V = \int d\rho \int z\rho d\theta.$$

The writer believes that the foregoing method could be used throughout the calculus, thus conserving the conception of the rate of change of a function as the basis of the entire subject. Most instructors would probably object to the elimination of the summation process on the ground that the latter is the most natural and vivid method of attacking practical problems. But is it not possible that the summation idea appears most natural to them only because they have been so long accustomed to it? To the beginner there is not very much logical clearness about the sudden jump from the inverted derivative notion to that of the integral as the limit of a sum. Certainly the former method presents no logical difficulties, and its retention throughout the calculus would at least add unity and consistency to the course.

The engineer is supposed to solve all his problems by the summation process. But after all, his method consists merely of setting an integral sign before the approximate value of a single element. Call this element an increment, and his integral reduces to the simple anti-derivative. For this intuitive process, the foregoing theorems would seem to provide a rigorous mathematical foundation.

ON SETTING UP A DEFINITE INTEGRAL WITHOUT THE USE OF DUHAMELS THEOREM.[1]

By Edward V. Huntington, Harvard University.

The purpose of this note is to state a simple theorem by means of which the ordinary process of "setting up an integral" may be simplified and made entirely rigorous without the use of Duhamel's Theorem or any of its modern substitutes. In view of the fundamental importance of the process of setting up an integral, it is hoped that such a simplification may be of value in both pure and applied mathematics.

To fix our ideas, let us take the familiar problem of finding the total attrac-

[1] This note contains the substance of two papers presented to the American Mathematical Society, April 29 and September 5, 1916, under the titles: (1) A simple example of the failure of Duhamel's Theorem, and (2) A simple substitute for Duhamel's Theorem.

tion, P, due to a thin rod, of length $b - a$, at a point O in line with the rod and at distance a from the nearer end. Suppose the linear density, ρ, of the rod to be variable, say $\rho = f(x)$, where $f(x)$ may be any function of x which is known for all values of x from $x = a$ to $x = b$. Also, let the law of attraction be taken in the general form, that is, suppose the attraction due to a particle to be proportional to $F(x)$ times the mass of the particle, where $F(x)$ may be any function of the distance x which is known from $x = a$ to $x = b$. [If the ordinary Newtonian law of the inverse square is adopted, then of course $F(x) = 1/x^2$.]

Now what is it that we actually do when we solve such a problem in practice? The actual steps are somewhat as follows:

First, we think of the rod as divided into small "elements," dx, where $dx = (b - a)/n$, and proceed to write down the attraction due to a typical element, say from $x = x$ to $x = x + dx$. Thus, the mass of the element is seen to be

$$f(x)dx$$

(at least approximately);[1] hence, the attraction at the point O due to the element is

$$kF(x)f(x)dx$$

(at least approximately),[2] where k is a factor of proportionality.

Next, having thus found the attraction due to a typical single element (at least approximately), we get the total attraction, P, due to all the elements, by simply writing an integral sign (with suitable limits) in front; thus:

$$P = \int_a^b kF(x)f(x)dx;$$

and in spite of the approximations used in setting up the integral, we feel assured that this final expression for P is *exact*.

Finally, we compute the value of this expression (whenever the law of attraction and the law of density are known), by the aid of a table of integrals or otherwise.

This is the simple, uncritical process of integration regarded as a method of summation. It is the process which is used probably more often than any other in the applications of the calculus.

The question that now presents itself is this: Under what conditions can this crude process be counted on to yield the correct result? Or, to put it in another way: Under what circumstances, if any, may the method be doubtful or even dangerous?

A sufficiently general answer to this question is exceedingly simple. Referring to our illustrative problem, *the naïve procedure just described will certainly be legitimate in the following case at least, namely, whenever the functions* $f(x)$ *and* $F(x)$

[1] This would be exact if the density throughout the element were the same as at its nearer end.

[2] This would be exact if all the attracting material in the element were concentrated at its nearer end.

are continuous. It is not necessary to consider any questions of "infinitesimals of higher order," or any questions of "uniformity"; the simple continuity of the two functions is sufficient.[1]

Stated in more general terms, we have, then, the following theorem, which appears to cover all or nearly all the cases that arise in practice:

THEOREM. *Suppose that a required quantity P is associated with a real interval, $x = a$ to $x = b$, in such a way that we are led to divide the interval into n small parts or "elements," Δx, and to regard P as the sum of n separate contributions, one from each element. Suppose also that a set of one or more functions, $F(x)$, $f(x)$, $\cdots$, can be found, such that, no matter what value of x is considered, and no matter how small Δx may be, the contribution from a typical element, $x = x$ to $x = x + \Delta x$, can be expressed "approximately" (see note 1) in the form*

$$[F(x)\,f(x)\;\cdots]\,\Delta x.$$

Then the required quantity P will be correctly given by the value of the definite integral

$$P = \int_a^b [F(x)f(x)\;\cdots]\;dx,$$

whenever the functions $F(x), f(x), \cdots$ are continuous from $x = a$ to $x = b$.

Note 1. The word "approximately" is here used in a technical sense, meaning that the exact value of the contribution in question lies between $[\bar{F}\cdot\bar{f}\cdots]\Delta x$ and $[\bar{\bar{F}}\cdot\bar{\bar{f}}\cdots]\Delta x$, where $\bar{F}, \bar{f}, \cdots$ are the smallest, and $\bar{\bar{F}}, \bar{\bar{f}}, \cdots$ the largest values of $F(x), f(x), \cdots$ in the element.

---

[1] The proof, which follows entirely familiar lines, is easily given, as follows. Let p be the attraction due to the part of the rod from $x = a$ to $x = x$, and let Δp be the added attraction due to the little additional element from $x = x$ to $x = x + \Delta x$. Then Δp will certainly lie between two extremes, namely:

$$k\bar{F}\bar{f}\Delta x \leqq \Delta p \leqq k\bar{\bar{F}}\bar{\bar{f}}\Delta x,$$

where $\bar{F}, \bar{f}$ are the smallest, and $\bar{\bar{F}}, \bar{\bar{f}}$ the largest values which $F(x)$ and $f(x)$ take on in the interval in question. But since $F(x)$ and $f(x)$ are continuous, each will take on at least once every value between its smallest and largest values in the interval; so that there will certainly exist values of x, say x' and x'', in the interval from $x = x$ to $x = x + \Delta x$, for which

$$\Delta p = kF(x')f(x'')\Delta x,$$

exactly. Now, keeping x fixed, divide through by Δx, and let Δx approach zero; then both x' and x'' will be squeezed down toward x as a limit, while $\Delta p/\Delta x$ approaches dp/dx, so that we have, in the limit,

$$dp/dx = kF(x)f(x), \qquad \text{whence,} \qquad p = \int kF(x)f(x)dx + C.$$

To determine the constant of integration, we have only to notice that $p = 0$ when $x = a$, so that

$$p = \int kF(x)f(x)dx - \left[\int kF(x)f(x)dx\right]_{x=a}$$

(the same value of the indefinite integral being used in each term). Finally, since P is the value of p when $x = b$, we have

$$P = \left[\int kF(x)f(x)dx\right]_{x=b} - \left[kF(x)f(x)dx\right]_{x=a} = \int_a^b kF(x)f(x)dx.$$

The proof is thus complete.

Note 2. In stating that P is to be regarded as the sum of n separate contributions, one from each element, we are assuming that P is the value, for $x = b$, of some function, $p(x)$, which has a definite value for each value of x from $x = a$ to $x = b$. It is not necessary to assume in advance that $p(x)$ is continuous, although it will, in fact, always be so whenever the other conditions of the theorem are satisfied.

Note 3. By an obvious modification, the theorem can be made to cover the case where the product $[F(x)\,f(x)\,\cdots]$ is replaced by any continuous combination of continuous functions.

With the statement of this theorem the main purpose of the present note is completed.

There are, however, a number of recent textbooks, notably Professor Osgood's *Calculus*, in which the process of setting up an integral as the limit of a sum is held to require, for complete rigor, the use of an auxiliary theorem known as Duhamel's Theorem.

In view of the simplicity and scope of the theorem proved above, the use of any auxiliary theorem such as Duhamel's would appear to be a superfluous complication. Moreover, Duhamel's Theorem in its ordinary form is known to be false.[1] Since the examples which have been adduced to prove this latter statement are rather complicated, the following simpler example may be of interest.

Duhamel's theorem in the primitive form in which it still appears in the textbooks, is as follows:

If $\alpha_1, \alpha_2, \cdots \alpha_n$ are a set of positive infinitesimals such that

$$\lim_{n=\infty} [\alpha_1 + \alpha_2 + \cdots + \alpha_n] = A;$$

and if $\beta_1, \beta_2, \cdots \beta_n$ are a second set of positive infinitesimals such that each β differs from the corresponding α by an infinitesimal of higher order, so that $\lim_{n=\infty} [\beta_i/\alpha_i] = 1$; then

$$\lim_{n=\infty} [\beta_1 + \beta_2 + \cdots + \beta_n] = A.$$

As our example, let $\alpha_i = 3/n$, and $\beta_i = (3n + 2i)/n^2$, where $i = 1, 2, \cdots n$. Then $\lim_{n=\infty} [\alpha_1 + \alpha_2 + \cdots + \alpha_n] = 3$; moreover, for any particular i, $\lim_{n=\infty} [\beta_i/\alpha_i] = \lim_{n=\infty} [1 + (2i)/(3n)] = 1$. The conditions of the theorem are therefore fulfilled, and according to the conclusion of the theorem, therefore, we should have $\lim_{n=\infty} [\beta_1 + \beta_2 + \cdots + \beta_n] = 3$. In fact, however, we have $\beta_1 + \beta_2 + \cdots + \beta_n = 3 + 2[1 + 2 + \cdots + n]/n^2 = 3 + (n + 1)/n$, so that $\lim_{n=\infty} [\beta_1 + \beta_2 + \cdots + \beta_n] = 4$.

[1] For a critical discussion of this theorem, see W. F. Osgood, "The integral as the limit of a sum and a theorem of Duhamel's," *Annals of Mathematics*, Ser. 2, Vol. 4 (1903), pp. 161–178; R. L. Moore, "On Duhamel's theorem," *ibid.*, Vol. 13 (1912), pp. 161–166; G. A. Bliss, "A substitute for Duhamel's Theorem," *ibid.*, Vol. 16 (1914), pp. 45–49. Further references may be found in the first of these papers. The present note is closely related to the third.

In view of this or similar examples,[1] it is clear that if Duhamel's Theorem is to be used at all, as a means of securing rigor, it must be taken in a modified form; interesting revised forms have in fact been proposed by Professors Osgood, R. L. Moore, and Bliss (*loc. cit.*); but, while these new forms leave nothing to be desired in point of rigor, none of them, as far as I know, has proved to be sufficiently simple to warrant its adoption in an elementary textbook.[2] The simplest plan to pursue in a first course in the calculus would therefore appear to be to omit Duhamel's Theorem altogether, substituting for it some such theorem as that suggested in the present paper.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Ruler and Compasses. By HILDA P. HUDSON. Longmans, Green and Company, London and New York, 1916. 143 pages.

This new volume of Longmans' Modern Mathematical Series is an attempt to collect from many sources solutions of problems and discussions of methods in which the Euclidean ruler and compasses are used as instruments; and to present them as part of a well-ordered development of the theory of such constructions. According to the author "the connecting link throughout the book is the idea of the whole set of ruler and compass constructions, its extent, its limitations, and its division."

The reader will require no more advanced mathematics than college algebra and elementary analytic geometry, although a knowledge of projective geometry and of the theory of equations in general will be helpful. The development of the theme is carefully carried out and there are no breaks in the logic, although in one or two places the author quotes a theorem which is developed later. This may prove a bit annoying to the reader with a minimum of preparation, but otherwise it is not a serious fault.

The subject matter of the text is presented as a whole in the introduction, which is rather well written, although it presupposes at times a rather full acquaintance with the material which is developed later. In Chapter II the criteria of possibility for ruler and compass constructions are established from the analytical point of view with the aid of a number of propositions from the elementary theory of equations. The chapter is divided into three parts; first, the constructions in which the ruler alone suffices, second, those in which the ruler and Euclidean compasses are required, and third, the construction of regular polygons of n sides. The cases in which n is a prime and n is composite

---

[1] During the discussion of this paper at the meeting of the Society, Professor D. Jackson suggested the following even simpler example: $\alpha_i = 1/n$ when $i \leqq n/2$, $\alpha_i = 0$ when $i > n/2$; $\beta_i = 1/n$.

[2] Even in Professor Osgood's own text (1907, revised edition 1909), the original (incorrect) form of Duhamel's theorem is retained, without comment. Professor Osgood's reasons for so doing may be found in his article of 1903 (*loc. cit.*).